Lectures on Perturbative QCD
or
from basic principles to current applications

Marco Stratmann
University of Regensburg
Outline of the lectures

**Lecture 1**: basic ideas; exploring the QCD final state

**Lecture 2**: origin of singularities; infrared safety

**Lecture 3**: QCD initial-state; factorization; renormalization

**Lecture 4**: more on factorization & renormalization; pdfs

**Lecture 5**: applications in hadron-hadron collisions; spin
Literature & useful links

Lecture notes & write-ups:

Wu-Ki Tung: *Perturbative QCD and the Parton Structure of the Nucleon*  
(from www.cteq.org)

Dave Soper: *Basics of QCD Perturbation Theory*  
(hep-ph/9702203)

J. Collins, D. Soper, G. Sterman: *Factorization of Hard Processes in QCD*  
(hep-ph/0409313)

CTEQ Collaboration: *Handbook of Perturbative QCD*  
(Rev. Mod. Phys. 67 (1995) 157 or from www.cteq.org)

Talks & lectures on the web:

annual CTEQ summer schools (tons of material !): www.cteq.org

1st summer school on QCD Spin Physics @ BNL: www.bnl.gov/qcdsp
Lecture 3

QCD initial state, partons, DIS factorization & renormalization
Electron-positron annihilation factorization (cont.)

let's see what factorization does for

\[ \begin{align*}
&\text{fragmentation functions } D_{\alpha}^h \\
&\text{contains all long-distance interactions} \\
&\text{hence not calculable but universal} \\
&\text{physical interpretation:} \\
&\text{probability to find a hadron carrying } \text{a certain momentum of parent parton} \\
&\text{hard scattering } \bar{F}_{\alpha} \\
&\text{contains only short-distance physics} \\
&\text{amenable to pQCD calculations}
\end{align*} \]
more explicitly we get

\[
\frac{d\sigma}{dz d\cos \theta} = \frac{\pi \alpha^2}{2s} \left[ F_T^A(x, Q)(1 + \cos^2 \theta) + F_L^A(x, Q) \sin^2 \theta \right]
\]

where

\[
F_{T,L}^A(z, Q) = \sum_a \hat{F}_{a,T,L}^A(z, Q) \otimes D_a^h(z, \mu_f)
\]

"convolution"

\[
f(x) \otimes g(x) \equiv \int_x^1 \frac{dy}{y} f \left( \frac{x}{y} \right) g(y)
\]

factorization scale (arbitrary!)
characterizes the boundary between short and long-distance physics
we postpone a closer look at the factorization scale dependence until we have introduced partons also in the initial state to cover electron-hadron and hadron-hadron interactions

the fact that experimental results do not depend on $\mu_f$ will then lead us to the famous DGLAP evolution equations
Deep-inelastic scattering (DIS) kinematics

let us now consider the process \( l(k) h(p) \rightarrow l'(k') + X \)

relevant kinematics:

- momentum transfer \( q^\mu = k^\mu - k'^\mu \)
  \[ Q^2 = -q^2 \]
- "Bjorken"-x
  \[ x = \frac{Q^2}{2p \cdot q} \]
- invariant mass
  \[ W^2 = (p + q)^2 = m_h^2 + \frac{1}{x} Q^2 \]
- fractional energy transfer
  \[ y = \frac{p \cdot q}{p \cdot k} \equiv \frac{E - E'}{E} \]

"deep-inelastic": \( Q^2 \gg 1 \text{ GeV}^2 \)

"scaling limit": \( Q^2 \to \infty, x \text{ fix} \)
Deep-inelastic scattering (DIS)

a neutral current event

here is how DIS looks like in "your" detector:

a neutral current event with photon-exchange
Deep-inelastic scattering (DIS) a charged current event

A charged current event with W-boson-exchange

(the electron turns into a neutrino which is "invisible")
Deep-inelastic scattering (DIS) towards the parton model

first analysis of DIS does not require any knowledge about QCD

electroweak theory tells us how the virtual vector boson couples:
(let's assume only photon exchange)

\[
\begin{align*}
d\sigma &= \frac{4\alpha^2}{s} \frac{d^3 \vec{k}'}{2|\vec{k}'|} \frac{1}{Q^4} L^{\mu\nu}(k, q) W_{\mu\nu}(p, q) \\
& \quad \text{phase space scat. lepton} \\
& \quad \text{photon propagator}^2 \\
& \quad \text{leptonic tensor} \\
& \quad \text{hadronic tensor contains information about hadronic structure}
\end{align*}
\]

(can be easily generalized to W/Z-boson exchange)
Deep-inelastic scattering (DIS) towards the parton model (cont.)

to all orders in the strong interaction $W_{\mu\nu}$ is given by the square of $\gamma^*(q) h(p) \rightarrow X$

symmetries (parity, Lorentz), hermiticity & current conservation tell us that

$$W_{\nu\mu} = W_{\mu\nu}^*$$
$$q_{\mu} W_{\mu\nu} = 0$$

$$W_{\mu\nu}(p, q) = - \left( g_{\mu\nu} - \frac{q_{\mu} q_{\nu}}{q^2} \right) F_1(x, Q^2)$$
$$+ \left( p_{\mu} - q_{\mu} \frac{p \cdot q}{q^2} \right) \left( p_{\nu} - q_{\nu} \frac{p \cdot q}{q^2} \right) \frac{1}{p \cdot q} F_2(x, Q^2)$$

structure functions

can be easily combined with the "trivial" leptonic tensor (just QED)

$$L_{\mu\nu} = 2(k^{\mu}k'^{\nu} + k^{\nu}k'^{\mu} - g^{\mu\nu} k \cdot k')$$
DIS cross section:

\[
\frac{d^2\sigma}{dx dQ^2} = \frac{4\pi\alpha^2}{Q^4} \left[ [1 + (1-y)^2]F_1(x, Q^2) + \frac{(1-y)}{x}[F_2(x, Q^2) - 2xF_1(x, Q^2)] \right]
\]

different y-dep. can differentiate between \(F_1\) and \(F_2 - 2xF_1\)

before we turn on the full glory of QCD dynamics
let's explore DIS in the naive quark-parton model:

\[\text{Bjorken scaling limit: } Q^2, \nu = p \cdot q \rightarrow \infty \text{ with } x \text{ fixed}\]

\[\bullet \text{ } F_1, F_2 \text{ obey scaling law, i.e., they are indep. of } Q^2\]
\[\bullet \text{ } \text{virtual photon scatters off pointlike constituents}\]
\[\bullet \text{ } \text{DIS is like taking a snapshot of the hadron}\]
Deep-inelastic scattering (DIS) space-time structure

this can be best understood in a reference frame where the proton moves very fast and $Q \gg m_h$ is big

(recall light-cone kinematics from Lecture 1)

<table>
<thead>
<tr>
<th>4-vector</th>
<th>hadron rest frame</th>
<th>Breit frame</th>
</tr>
</thead>
<tbody>
<tr>
<td>$(p^+, p^-, \bar{p}_T)$</td>
<td>$\frac{1}{\sqrt{2}}(m_h, m_h, \bar{0})$</td>
<td>$\frac{1}{\sqrt{2}}(\frac{Q}{x}, \frac{xm_h^2}{Q}, \bar{0})$</td>
</tr>
<tr>
<td>$(q^+, q^-, \bar{q}_T)$</td>
<td>$\frac{1}{\sqrt{2}}(-m_h x, \frac{Q^2}{m_h x}, \bar{0})$</td>
<td>$\frac{1}{\sqrt{2}}(-Q, Q, \bar{0})$</td>
</tr>
</tbody>
</table>

Lorentz boost

in general $(a^+, a^-, \bar{a}_T) \rightarrow (e^{\omega} a^+, e^{-\omega} a^-, \bar{a}_T) = (a'^+, a'^-, \bar{a}')$

here: $e^{\omega} = Q/(xm_h)$
Deep-inelastic scattering (DIS) space-time structure (cont.)

simple estimate for typical time-scale of interactions among the partons inside a fast-moving hadron:

rest frame: \( \Delta x^+ \sim \Delta x^- \sim \frac{1}{m} \)

Breit frame: \( \Delta x^+ \sim \frac{1}{m m} = \frac{Q}{m^2} \) large

\( \Delta x^- \sim \frac{1}{m Q} = \frac{1}{Q} \) small

interactions between partons are spread out inside a fast moving hadron

How does this compare with the time-scale of the hard scattering?
Deep-inelastic scattering (DIS) space-time structure (cont.)

now let the virtual photon meet our fast moving hadron ...

upshot:
• partons are free during the hard interaction
• hadron effectively consists of partons that have momenta $(p_i^+, p_i^-, p_i^z)$
• convenient to introduce momentum fractions $0 < \xi_i \equiv p_i^+/p^+ < 1$
Deep-inelastic scattering (DIS) according to the naive parton model

the space-time picture suggests the possibility of separating short and long-distance physics (=factorization!)

turned into the language of Feynman diagrams DIS looks like

\[
\frac{d^2\sigma}{dx dQ^2} \sim \int_0^1 d\xi \sum_a f_a^h(\xi) \frac{1}{\xi} \left. \frac{d^2\hat{\sigma}}{d\hat{x} dQ^2} \right|_{\hat{x} = x/\xi} + O\left(\frac{m}{Q}\right)
\]

- \( f_a^h(\xi) d\xi \) probability to find a parton with flavor \( a \) in a hadron \( h \) carrying light-cone momentum \( \xi \mathbf{p}^+ \)
- \( d^2\hat{\sigma}/d\hat{x} dQ^2 \) cross section for electron-parton scattering
Deep-inelastic scattering (DIS) naive parton model (cont.)

let's see how the **scaling property** comes about:

a quick computation of the hard scattering cross section at LO yields:

\[
\frac{d\hat{\sigma}}{dQ^2} = \frac{2\pi \alpha_s^2 e_q^2}{Q^4}[1 + (1 - y)^2]
\]

the scattered quark is on-mass shell:

\[
\xi \cdot p^+ + q^+ = 0 \quad \text{implies that } \xi = x \text{ at LO}
\]

\[
\frac{d^2\hat{\sigma}}{dx dQ^2} = \frac{4\pi \alpha_s^2}{Q^4} \frac{1}{2}[1 + (1 - y)^2] e_q^2 \delta(x - \xi)
\]
Deep-inelastic scattering (DIS)  
aive parton model (cont.)

compare with the definition of the structure functions!

find:  
\[ F_2(x) = 2x F_1(x) = \sum_{a=q,\bar{q}} \int_0^1 d\xi f_a(\xi) x e_a^2 \delta(x - \xi) \]
\[ = \sum_{a=q,\bar{q}} e_a^2 x f_a(x) \]

- the desired scaling property: no dependence on \( Q^2 \)
- \( F_L(x) \equiv F_2(x) - 2x F_1(x) \) vanishes! (Callan-Gross relation)
  (test that quarks are spin-1/2: they cannot absorb a long. pol. \( \gamma^* \))

How does this compare with experiment?
Deep-inelastic scattering (DIS)
SLAC-MIT experiment of 1969

two unexpected results:

partons!

scaling!

birth of the pre-QCD parton model
Deep-inelastic scattering (DIS)  
HERA: scaling violations

the first (and only) ep-collider:

observe strong scaling violations

What does pQCD has to say about this?
Deep-inelastic scattering (DIS) according to pQCD

we got a long way (parton model) without invoking QCD

now we have to study QCD dynamics in DIS
- this leads to similar problems already encountered in $e^+e^-$

let's try to compute the QCD corrections to the parton model picture

\[ \alpha_s \text{ corrections to the LO process} \quad \text{photon-gluon fusion} \]

our experience so far: have to expect divergencies!

we cannot calculate with infinities \( \rightarrow \) introduce some regulator
remove it in the end
Deep-inelastic scattering (DIS) according to pQCD (cont.)

possible regulators:

- small quark/gluon masses
  intuitive and transparent
  but works only in NLO

- dimensional regularization
  = change dimension of space-time to 4-2\(\epsilon\)
  calculations more involved; works in general

let's choose this one

depending on your choice singularities will be hidden as

large logarithms, e.g., \(\log(m^2/Q^2)\) or as \(1/\epsilon\)

only if we have done everything consistently, including factorization,
we can safely remove the regulator and can compare to experiment
Deep-inelastic scattering (DIS) according to pQCD (cont.)

the general structure of the $\alpha_s$ corrections looks like this:

$$\frac{d^2\hat{\sigma}}{dx dQ^2}|_{F_2} \equiv \hat{F}_2^q$$

$$= e_q^2 x \left[ \delta(1 - x) + \frac{\alpha_s(\mu_r)}{4\pi} \left[ P_{qq}(x) \ln \frac{Q^2}{m_q^2} + C_2^q(x) \right] \right]$$

$$= \sum_{q} e_q^2 x \left[ 0 + \frac{\alpha_s(\mu_r)}{4\pi} \left[ P_{qg}(x) \ln \frac{Q^2}{m_q^2} + C_2^g(x) \right] \right]$$
the structure of the results

large logarithms + finite coefficients

already hints towards factorization ...

fasten your seatbelts and prepare for
the "magic" of factorization
Deep-inelastic scattering (DIS) according to pQCD (cont.)

first it is important to notice that

- large logarithms (or $1/\varepsilon$) incorporate all long-distance physics (collinear emission)
- the coefficients $P_{ij}(x)$ multiplying the log's are universal and calculable (splitting functions)

the physical meaning of the splitting functions is easy:

$P_{ij}(x)$: probability that a parton $j$ splits collinearly into a parton $i$ (and something) carrying a momentum fraction $x$
Deep-inelastic scattering (DIS) according to pQCD

to obtain the physical cross section we have convolute our partonic results with the parton densities (like in the parton model!):

e.g. for

\[ F_2(x, Q^2) = x \sum_{a=q, \bar{q}} e_q^2 \left[ f_{a,0}(x) + \frac{\alpha_s}{2\pi} \int_x^1 \frac{d\xi}{\xi} \right] \]

\[ f_{a,0}(x) \left[ P_{qq} \left( \frac{x}{\xi} \right) \ln \frac{Q^2}{m_g^2} + C_g^q \left( \frac{x}{\xi} \right) \right] \]

now ... here comes the "trick":

the \( f_{a,0}(x) \) are unmeasurable bare (=infinite) densities and need to be re-defined (=renormalized) to make them physical
Deep-inelastic scattering (DIS) according to pQCD

The renormalized quark densities (at order $\alpha_s$) are given by:

$$f_q(x, \mu_f^2) \equiv f_{q,0}(x) + \frac{\alpha_s}{2\pi} \int_x^1 \frac{d\xi}{\xi} f_{q,0}(\xi) P_{qq} \left( \frac{x}{\xi} \right) \ln \left( \frac{\mu_f^2}{m_q^2} \right) + z_{qq}$$

Absorb all long-distance singularities at a factorization scale $\mu_f$ into $f_{q,0}$

Physical densities: not calculable in pQCD but universal

Insert back and keep only terms up to $\alpha_s$:

$$F_2(x, Q^2) = x \sum_{a=q,\bar{q}} e_q^2 \int_x^1 \frac{d\xi}{\xi} f_a(\xi, \mu_f^2)$$

$$\left[ \delta(1 - \frac{x}{\xi}) + \frac{\alpha_s(\mu_f)}{2\pi} \left[ P_{qq} \left( \frac{x}{\xi} \right) \ln \frac{Q^2}{\mu_f^2} + (C^q - z_{qq}) \left( \frac{x}{\xi} \right) \right] \right]$$

This is our final result!
Let's analyze it piece by piece!
Deep-inelastic scattering (DIS) according to pQCD

The physical structure function is independent of $\mu_f$ (this will lead to the concept of renormalization group eqs.)

Both, pdf's and the short-dist. coefficient depend on $\mu_f$ (choice of $\mu_f$: shifting terms between long- and short-distance parts)

$$ F_2(x, Q^2) = x \sum_{a=q, \bar{q}} e_a^2 \int_x^1 \frac{d\xi}{\xi} f_a(\xi, \mu_f^2) $$

$$ \left[ \delta(1 - \frac{x}{\xi}) + \frac{\alpha_s(\mu_r)}{2\pi} \left[ P_{qq} \left( \frac{x}{\xi} \right) \ln \frac{Q^2}{\mu_f^2} + (C_2^q - z_{qq}) \left( \frac{x}{\xi} \right) \right] \right] $$

Yet another scale: $\mu_r$ due to the renormalization of ultraviolet divergencies

Short-distance "Wilson coefficient"

Choice of the factorization scheme
that was a lot of material and perhaps hard to swallow

let us postpone questions like

• What the hell does renormalization?
• Pdfs are universal, so what is their formal definition?
• What should I do with all these arbitrary scales?
• What is a factorization scheme?

until tomorrow!